A CONTRIBUTION TO THE THEORY OF PARAMETRIC EXCITATION OF THERMOMECHANICAL OSCILLATIONS

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The conditions for a rather fast increase in parametric thermomechanical oscillations have been revealed by using numerical solutions of some types of Hill's equation. It is shown that the occurrence of a real parametric resonance is essentially determined by the character of thermal modulations. It is hypothesized that the indicated trends are observed in oscillations of any physical nature.

In present-day thermal physics the studies of thermomechanical oscillations (TMO) of thin heaters have received attention with a view of enhancing heat transfer into the ambient medium [1, 2]. An important feature of such oscillations is their proclivity to self-excitation owing to parametric amplification of random small vibrations or temperature oscillations.

When applying TMO in practice or thermotechnical calculations, use is made of the so-called classical parametric resonance (PR) theory which is expounded in special literature [3-6]. However, as is shown in what follows, the theory cannot give an answer to a number of questions arising when attempting to employ the parametric TMO in thermal power engineering.

The point is that the above theory, which relies basically on the works of Stratt and Ains [7, 8], determines the conditions under which PR may originate in principle if modulations follow the harmonic law (to this case there corresponds Mathieu's equation [9]) or the meander law (Meissner's equation [5]). In practice different types of modulations can, however, be encountered. On the other hand, in the case of parametric TMO's the temperature fluctuations acquire a complex relaxational character because of the thermal inertia of heaters, and the modulations of the heated wire tension σ generated by these fluctuations are still more complex. The classical theory does not consider the conditions for the occurrence of PR in such cases.

Another deficiency of the theory is that it has not revealed the dependence of the value of the characteristic exponent μ (which determines the rate of parametric pumping) on the resonance order n, initial phase shift $\Delta\varphi$, and on other factors in the case of different types of modulating oscillations. Therefore, the theory cannot indicate techniques for computing the rate of increase in the amplitude of oscillations in each specific case.

At the same time, this problem is extremely important in practice: if under certain conditions the oscillational energy increases at a negligible rate, then, even though PR takes place in principle, the amplitude will become appreciable after an unusually great interval of time. From now on, we shall say that there is no real parametric resonance (RPR) in this case.

In light of what has been said, we decided to determine the conditions for the onset of RPR when its application in technology becomes expedient. For the subsequent presentation to be clearer, we shall briefly formulate the essence of the problem using the example of a spring pendulum whose equation of free oscillations has the form

$$m\ddot{x} + kx = 0, \tag{1}$$

The solution of this equation is $x = A\sin\omega_0 t$, where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the pendulum.

If the energy-intensive parameters of the system, i.e., mass m and elasticity of the pendulum k, vary with time, the oscillational energy ceases to be a constant value, and its time derivative becomes equal to

$$\frac{dE}{dt} = \frac{x^2}{2}\frac{dk}{dt} - \frac{\dot{x}^2}{2}\frac{dm}{dt}\,. \tag{2}$$

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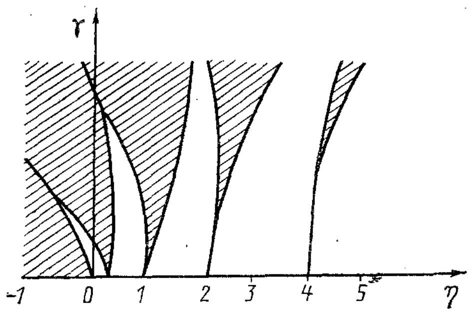

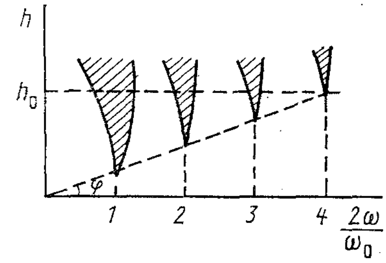

Fig. 1 Fig. 2

Fig. 1. Stability diagram for Mathieu's differential equation. Dashed lines isolate the parametric excitation instability regions.

Fig. 2. Instability regions for a dissipative system ($\tan\varphi = 2\delta$).

In the case of periodic variations in these parameters, the movement of the pendulum is already described by Hill's equation

$$\ddot{x} + \psi(t)x = 0, \tag{3}$$

where ψ(t) is a periodic function whose period is $T = 2\pi/\omega$, ω and is the frequency of modulations.

As shown in [10], the solution of Hill's equation can be presented in the form

$$x(t) = \exp \mu t \Phi(t). \tag{4}$$

Here Φ(t) is a certain periodic function with the same period T. The quality μ in Eq. (4) is called the characteristic exponent. Its value determines the solution stability: when $\mu < 0$, the oscillations of x(t) are attenuating, tending to zero when $t \to \infty$ and the solution is asymptotically stable; on the other hand, when $\mu > 0$, then with $t \to \infty$ the quantity x(t) increases infinitely and the solution is unstable.

In the above-mentioned works [7, 8] the conditions were found under which the exponent μ reduces to zero and the solution becomes purely periodic. The curves $\mu = 0$ separate the stability and instability regions (it is only in the latter region that parametric excitation of oscillations is possible).

It should be emphasized that in a general form the rigorous solution of the indicated problem for Hill's equation (3) has not been obtained up to now. In [7, 8] functional series were obtained that satisfy Eq. (3) for the particular cases:

(1) Mathieu's equation in which ψ(t) takes on the form

$$\psi(t) = \psi_0 + \Delta\psi \cos \omega t; \tag{5}$$

(2) Meissner's equation (for meander-type modulations)

$$\psi(t) = \psi_0 + \Delta\psi \,\mathrm{sign} \cos \omega t. \tag{6}$$

In Eqs. (5) and (6) the quantity ψ_0 represents the square of the natural frequency ω_0^2 (ω is the frequency of modulations, and $\Delta\psi$ is the absolute depth of modulation).

Introducing the notation $\tau = \omega t$, $\eta = \psi_0/\omega^2$, and $\gamma = \Delta\psi/\omega^2$, we shall reduce Mathieu's equation (5) to the form usually used in the PR theory [5]

$$\ddot{\psi} + (\eta + \gamma \cos \tau)\psi(\tau) = 0. \tag{7}$$

Numerical solution of this equation allowed Ains and Stratt to obtain a set of the curves $\mu(\eta, \gamma) = 0$ depicted in Fig. 1. It is seen from the figure that in the presence of modulations ($\gamma > 0$), instability regions ($\mu > 0$) originate whose width increases with γ. Conversely, when $\gamma \to 0$ these regions reduce and contact the abscissa axis at the points $\eta = (n/2)^2$ (n = 1, 2,...). From this it follows that for the occurrence of PR (in the case of small modulations $\gamma \approx 0$) the condition of frequencies should be satisfied

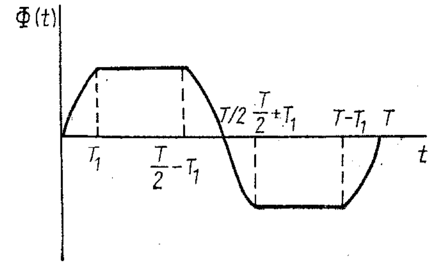

Fig. 3. Graph of Φ(t).

$$\omega \simeq \frac{2}{n}\,\omega_0, \tag{8}$$

which can be often put in equivalent form

$$\omega = \frac{2}{n}\,\omega_0 + \varepsilon, \tag{9}$$

where the "unbalance" of ε becomes vanishingly small when $\gamma \to 0$.

Analysis of Fig. 1 also shows that the instability regions expand ($\varepsilon \sim h$) with the growth of the relative depth of modulation $h = \Delta\psi/\psi_0 = \gamma/\eta$, whereas with an increase in the resonance order n, they, conversely, contract ($\varepsilon \sim h^n$, $h << 1$), with the value of the exponent μ also decreasing [10].

Based on the classical theory of PR, the effect of dissipative forces on the conditions for the occurrence of resonance was also investigated and it was found that in this case the solution instability appeared only in the event when the modulation depth exceeded a certain "threshold" value h^0 (for the first resonance $n = 1$ this threshold is $h_1^0 \approx 4\delta/\omega_0$, where δ is the extinction coefficient of oscillations). As the number n increases, the threshold height h_n^0 rises directly with n. Thus, in dissipative systems the instability regions displace upward following a linear law (the inclination angle of the straight line along which the tips of the instability regions lie is determined by the value of δ; see Fig. 2).

When evaluating the results obtained, one should take note of a number of fundamental conclusions obtained with the aid of the classical theory of PR, but not all of the problems associated with the parametric excitation of oscillations have been studied; moreover, some of the physical and mathematical experiments conducted at our laboratory led to results not consistent with the PR theory. In our experiments the oscillational parameters were modulated at the expense of corresponding temperature fluctuations.

The considerations given above led us to mathematically simulate the process of the parametrical excitation of TMO and to use a fast computer for investigating the dependence of the exponent μ on the character of temperature fluctuations $\theta(t)$ and initial phase shift $\Delta\varphi$. Such studies will aid in formulating the conditions for the occurrence of RPR, which is essential for practical thermal engineering and with which thermomechanical oscillations of a heater will take place at a much greater rate.

Limiting ourselves to small modulation depths h, we elucidated those frequency intervals ω/ω_0 and initial phase shifts $\Delta\varphi$ at which the exponent μ is markedly greater than zero, so that the TMO amplitudes already attain appreciable values after several tens of periods.

With the aid of our special programs composed for an EC-1061 computer we managed to solve the problem posed for several types of thermal modulations and to obtain the following results.

I. The character of the parametric amplification of TMO in the case when temperature modulations follow the harmonic law and their depth is small ($h = 0.01$-0.2) was determined. The oscillations increase markedly near the frequency $\omega \approx 2\omega_0$. This means that only the first-order resonance is observed ($n = 1$); for all the remainder $n = 2, 3, \ldots$, the exponent μ is negligibly small and there is virtually no pumping of energy.

II. A similar investigation for meander-type modulations shows that in this case the RPR practically occurs only near odd values of $n = 1, 3, 5, \ldots$, the exponent μ decreases monotonically with the increase in the odd number. In the even instability regions (at small modulation depths h) the exponent μ, while being positive, turns to be a negligible value.

III. A family of periodic functions were considered which vary continually and smoothly from a sinusoid to a meander with an invariant period $T = 2\pi/\omega$. One such function is presented in Fig. 3. Analytically this family is defined as

$$\Phi(t) = \begin{cases} \sin\frac{\pi}{2}\frac{t}{T_1} & (0 \leqslant t \leqslant T_1), \\ 1 & (T_1 \leqslant t \leqslant T/2 - T_1), \\ \cos\frac{\pi}{2}\frac{t + T_1 - T/2}{T_1} & (T/2 - T_1 \leqslant t \leqslant T/2). \end{cases} \quad (10)$$

In the second half-period, the function $\Phi(t)$ changes sign. Varying the value of the parameter T_1 from 0 to T/2, we obtain an infinite set of functions of the indicated family. Assuming that temperature modulations follow the law expressed by Eq. (10), the character of the parametric amplification of TMO can be determined by solving the corresponding Hill's equation

$$\ddot{x} + \omega_0^2 [1 + h\Phi(t)]\, x = 0. \quad (11)$$

The computer numerical solution allowed us to elucidate the dependence of the exponent μ on the specific type of modulating function, i.e., on the ratio T_1/T (at small modulation depths h = 0.05-0.2). It has turned out that the rate of parametric increase in TMO is practically significant under the following conditions:

1. When $T_1/T = 0$, i.e., modulations are described by Meissner's function (6), when the frequency ω of temperature fluctuations is equal to $\omega \simeq 2\omega_0/(2k - 1)$ (k = 1, 2,...).

2. When $T_1/T = 1$ (modulations are harmonic), only at one frequency ω of the periodic variations of temperature, namely $\omega \simeq 2\omega_0$.

3. A monotonic increase in the ratio T_1/T from 0 to 1 is accompanied by a tendency toward the rise in the parametric pumping rate, but then the exponent μ will attain marked values (decreasing with the number n) in odd instability regions, whereas in even regions the values of μ remain virtually insignificant (for not very large h).

4. The character of the parametric TMO and, in particular, the value of the exponent μ is affected by the initial phase shift $\Delta\varphi$ between modulating and basic oscillations. In the case of harmonic modulations the relative increment of the system energy for the period turns to be proportional to $\sin(\Delta\varphi)$ and also to the modulation depth h: $\Delta E/E \sim h \sin(\Delta\varphi)$. (This suggests that usually $\Delta\varphi = \pi/2$ in the case of harmonic modulations.) But when modulations follow the meander type with the frequency

$$\omega = \frac{2\omega_0}{2k - 1} + \varepsilon,$$

peculiar beatings are observed in the system, i.e., the amplitudes of the developed TMO's vary with the period $\tau = 1/\varepsilon$. Such parametric beatings were observed in our experiments [11] whose formation mechanism is described briefly in [2].

CONCLUSIONS

1. To apply the parametric excitation of the thermomechanical oscillations of heat-emitting surfaces practically in engineering, it is necessary that for each system of temperature modulations one could select those conditions for the onset of PR which would ensure the occurrence of RPR.

2. The conclusions drawn from the analysis of the parametric thermomechanical oscillations can be extended to vibrational systems of arbitrary physical nature.

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